

POSITIVITY QUESTIONS IN KÄHLER-EINSTEIN THEORY

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ABSTRACT. Tian initiated the study of incomplete Kähler-Einstein metrics on quasi-projective varieties with cone-edge type singularities along a divisor, described by the cone-angle $2\pi(1 - \alpha)$ for $\alpha \in (0, 1)$. In this paper we study how the existence of such Kähler-Einstein metrics depends on α . We show that in the negative scalar curvature case, if such Kähler-Einstein metrics exist for all small cone-angles then they exist for every $\alpha \in (\frac{n+1}{n+2}, 1)$, where n is the dimension. We also give a characterization of the pairs that admit negatively curved cone-edge Kähler-Einstein metrics with cone angle close to 2π . Again if these metrics exist for all cone-angles close to 2π , then they exist in a uniform interval of angles depending on the dimension only. Finally, we show how in the positive scalar curvature case the existence of such uniform bounds is obstructed.

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1. INTRODUCTION

G. Tian [Tia96] outlined a very general program for the construction of *incomplete* Kähler-Einstein metrics on quasi-projective varieties $X \setminus D$ where X is a smooth projective variety and $D \subset X$ is a simple normal crossing divisor. Such metrics have cone-edge like singularities along D , their asymptotic behavior is described by the cone-angle $2\pi(1 - \alpha)$ for $\alpha \in (0, 1)$. These metrics have been extensively studied, see for example [Tia96], [Don10], [Ber11], [Bre11], [CGP11], [JMR11], [MR12]. Recently, Jeffres-Mazzeo-Rubinstein [JMR11] and Mazzeo-Rubinstein [MR12] made substantial progress toward the completion of Tian's original program. In particular, they show that such metrics, with negative scalar curvature and cone angle $2\pi(1 - \alpha)$, exist when $K_X + \alpha D$ is ample, see also [CGP11] for the case $\alpha \in [\frac{1}{2}, 1)$.

In this paper we consider two questions motivated by these results.

- Characterize those pairs (X, D) for which a Kähler-Einstein metric with cone angle $2\pi(1 - \alpha)$ exists for some α .

- Given (X, D) , describe the values α for which such a metric exists.

We have a satisfactory answer to these questions for small cone-angles in the negative scalar curvature case.

Theorem 1.1. *Let X be a smooth, n -dimensional projective variety and $D \subset X$ a simple normal crossing divisor. The following are equivalent.*

- (1) (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for all $\alpha \in (1 - \epsilon, 1)$ and some $\epsilon > 0$.
- (2) (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for every $\alpha \in (\frac{n+1}{n+2}, 1)$.
- (3) The self-intersection $(K_X + D)^n$ is positive, $(K_X + D) \cdot C \geq 0$ for every curve $C \subset X$ and there are no curves $C \subset X$ such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.

Similarly, we can treat the negative scalar curvature case for cone-angles close to 2π .

Theorem 1.2. *Let X be a smooth, n -dimensional projective variety of general type and $D \subset X$ a simple normal crossing divisor. The following are equivalent.*

- (1) (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for all $\alpha \in (0, \epsilon)$ and some $\epsilon > 0$.
- (2) (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for every $\alpha \in (0, \frac{1}{2n+1})$.
- (3) $K_X \cdot C \geq 0$ for every curve $C \subset X$ and there are no curves $C \subset X$ such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$.

The proofs of these results rely on the cone theorem and on the base point free theorem. For the technical details see Section 4.

Remarkably, in the positive scalar curvature case the existence of such uniform bounds is obstructed for both small and large cone-angles, see Examples 6.2 and 6.11. Nevertheless, we are able to quantify this obstruction when $-K_X - D$ and $-K_X$ are ample. For small cone-angles we obtain that this obstruction depends *only* on the self-intersection $(-K_X - D)^n$, compare with Example 6.2. More precisely, we can state the following.

Theorem 1.3. *Let \mathfrak{B} be a set of pairs (X, D) such that X is a smooth projective variety of dimension n and D is a reduced effective divisor with simple normal crossing support such that $-(K_X + D)$ is ample. The following are equivalent.*

- (1) There exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_0, 1]$ and any $(X, D) \in \mathfrak{B}$.
- (2) There exists a positive integer M_0 such that $(-K_X - D)^n \leq M_0$ for any $(X, D) \in \mathfrak{B}$.

We can also treat the case of cone-angles close to 2π . In this case the obstruction depends *only* on the intersection number $D \cdot (-K_X)^{n-1}$, compare with Example 6.11.

Theorem 1.4. *Let \mathfrak{D} be a set of pairs (X, D) such that X is a smooth projective Fano variety of dimension n and D is a reduced effective divisor with simple normal crossing support. The following are equivalent.*

- (1) *There exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in [0, \alpha_0)$ and any $(X, D) \in \mathfrak{D}$.*
- (2) *There exists a positive integer M_0 such that $D \cdot (-K_X)^{n-1} \leq M_0$ for any $(X, D) \in \mathfrak{D}$.*

The strategies used in the proofs of Theorems 1.3 and 1.4 are completely different from the techniques employed in the negative scalar curvature case. The proofs of these theorems rely on effective very ampleness results as well as on boundness of Fano varieties. For the technical details see Section 6.

We now recall some of the key features of Tian's program [Tia96] and their relation with the theme of positivity in algebraic geometry. Incomplete Kähler-Einstein metrics are geometrically very interesting. Unlike the complete case, they have a much more interesting moduli theory as they are not subject to Yau's generalized Schwarz lemma. Furthermore, in Tian's original program [Tia96], they should also be used to better understand the complete Kähler-Einstein metrics previously constructed on X and $X \setminus D$. Recall that T. Aubin and S.-T. Yau constructed negatively curved Kähler-Einstein metrics on X when K_X is ample, see [Aub78] and [Yau78a]. Similarly, there are many existence results for complete negatively curved Kähler-Einstein metrics on $X \setminus D$ under certain positivity assumption for $K_X + D$, see for example [TY87], [Yau78b], [Wu08], [Wu09]. For precise definitions and details see Section 2. Now, one should be able to recover these metrics by taking limits of Kähler-Einstein metrics with cone-edge singularities as the cone angle goes to 2π or 0. Moreover, the study of this limiting behavior should give a better understanding of the asymptotic behavior of the complete Kähler-Einstein metrics on $X \setminus D$.

Kähler-Einstein metrics with cone-edge singularities are also very interesting from a *positivity* point of view. Their existence implies strong positivity properties for \mathbb{R} -divisors of the form $K_X + \alpha D$ or $-(K_X + \alpha D)$ for some $\alpha \in (0, 1)$. We refer to Section 2 for the precise definitions, results and to Proposition 2.10 for the necessary condition for the existence of such metrics. Thus, the existence of these metrics appears to be associated with the positivity of the log-canonical or anti-log-canonical divisor of a Kawamata log-terminal (klt) pair rather than a log-canonical (lc) pair. This fact turns out to be extremely useful since the *base-point free* theorem is currently known for big, nef and klt pairs only. We exploit systematically this point in Section 4. For the definitions of klt and lc pairs, the statement of the base-point free theorem and the results from algebraic geometry used in the rest of this work we refer to Section 3.

To sum up, it is important to study which positivity properties are preserved, lost or gained by passing from a lc pair (X, D) to a klt pair $(X, \alpha D)$, $\alpha \in (0, 1)$, and viceversa. We find that these questions can be approached by studying the

behavior of certain positivity thresholds, see Sections 4, 5 and 6. For example, given a pair (X, D) with $K_X + D$ big and nef the key object to study is the so called *nef threshold* $r(X, D)$, see Definition 4.2 and the results in Section 4. All the questions we address here are explicitly formulated throughout the text, see Questions 4.1, 4.9, 6.1 and 6.10. We provide complete solutions to all of them through Theorems 4.15, 4.29, Example 6.2 and Theorems 1.3, 1.4. These results seem to have interest of their own. Moreover, they have applications beyond the study of Kähler-Einstein metrics.

The paper is organized as follows. In Section 2, we recall the theory of Kähler-Einstein metrics on quasi-projective varieties. We discuss the existence of both complete and incomplete Kähler-Einstein metrics of negative and positive scalar curvature. It is shown how the existence of such metrics motivates a number of natural positivity questions. These questions are thoroughly studied in Sections 4, 5 and 6. In Section 3, we state some fundamental results from the theory of the minimal model which we repeatedly use in this work. In Section 5, we study the notion of ampleness for line bundles on quasi-projective varieties. This analysis is important for the study of negatively curved Kähler-Einstein metrics beyond the cone-edge asymptotic, see Corollary 5.11 and the remarks which follows. Finally, all the results are motivated and supported with examples.

2. KÄHLER-EINSTEIN METRICS ON QUASI-PROJECTIVE VARIETIES

In this section we recall and collect some generalities regarding the existence of both complete and incomplete Kähler-Einstein metrics on quasi-projective varieties. In particular, we discuss the *necessary* conditions for the existence of such metrics. This analysis uses the language and the basic theory of *Kähler currents* on Kähler manifolds [Dem01]. The results obtained motivate the positivity questions addressed in the rest of this work.

Let us start by fixing notation and by giving some definitions. Let X be a n -dimensional projective manifold and $D = \sum_i D_i$ be a reduced simple normal crossing divisor. Concretely, this simply means that the irreducible complex hypersurfaces D_i are smooth and that they intersect transversally. Given the pair (X, D) , let us recall the notion of a Kähler metric with Poincaré type singularities along D . A smooth Kähler metric $\hat{\omega}$ on $X \setminus D$ is said to have Poincaré singularities along the D_i 's if for any point $p \in D$ and coordinate neighborhood $(\Omega; z_1, \dots, z_n)$ centered at p for which

$$D|_{\Omega} = \{z_1 \cdot \dots \cdot z_k = 0\}$$

then $\hat{\omega}$ is *quasi-isometric* in Ω to the following model metric

$$(1) \quad \omega_0 = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right).$$

It is easy to construct Kähler metrics with Poincaré singularities along a divisor.

Example 2.1. Let ω be a Kähler metric on X . For simplicity let D be a smooth reduced divisor. Choose an Hermitian metric $\|\cdot\|$ on $\mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{O}(D))$ be a defining section for the divisor D . Then for $T > 0$ big enough

$$\hat{\omega} = T\omega - \sqrt{-1}\partial\bar{\partial} \log \log^2 \|s\|^2$$

defines Kähler metric on $X \setminus D$ with Poincaré singularities along D .

Note that any metric on $X \setminus D$ which is quasi-isometric to the model given in 1 is necessarily a complete Riemannian metric. In fact, it suffices to show that the boundary divisor D is “metrically at infinity”, which is clearly the case here since

$$\int_0^l \frac{dr}{r|\log r|} = -\log \log \frac{1}{r} \Big|_0^l = \infty.$$

Moreover, such metrics have finite volume as one can easily see computing

$$\int_0^{2\pi} \int_0^l \frac{rdrd\theta}{r^2(\log r)^2} < \infty.$$

If we now restrict our attention to Kähler metrics on $X \setminus D$ which have Poincaré singularities along D then they can also be regarded as *global* objects on X as *currents*. For the basic definitions and properties of complex currents we refer to the books [GH78] and [Dem01].

Proposition 2.2. Let $\hat{\omega}$ be a Kähler metric on $X \setminus D$ with Poincaré singularities along D . Then $\hat{\omega}$ defines a Kähler current on X .

Proof. Recall that in order to extend a positive closed $(1, 1)$ -current across an analytic set it suffices to check that it has finite mass near it. For a proof of this important fact see Théorème 1.1 in [Sib85] and the bibliography there. Now, this amounts to the fact that the Poincaré metric has finite volume. Finally, $\hat{\omega}$ defines a Kähler current since the condition given in 1 ensures that it dominates a positive $(1, 1)$ -form on X . \square

Remark 2.3. It seems that the idea of bounding forms on quasi-projective variety by Poincaré type metrics goes at least back to Cornalba and Griffiths [CG75]. Moreover, many of the extension properties of such forms were also explained by Mumford [Mum72].

We now return to the existence problem for Kähler-Einstein metric with Poincaré type singularities. Note that, classical theorems in global Riemannian geometry [Pet06], can be used to rule out the existence of such Einstein metrics when the scalar curvature is non-negative.

It is easy to construct Kähler-Einstein metrics with Poincaré singularities along a divisor and negative scalar curvature.

Example 2.4. Let Σ be a finite volume hyperbolic Riemannian surface and let $\bar{\Sigma} = \Sigma \cup \{p_1, \dots, p_k\}$ be its natural compactification. Then the hyperbolic metric on Σ is a Kähler-Einstein metric with Poincaré singularities along the p_i ’s.

Note that, in Example 2.4, the *logarithmic* canonical bundle

$$K_{\bar{\Sigma}} + P, \quad P := \{p_1, \dots, p_k\},$$

associated to the pair $(\bar{\Sigma}, P)$ is an *ample* divisor on $\bar{\Sigma}$. Recall that Σ admits a hyperbolic metric if and only if $2g(\bar{\Sigma}) - 2 + k > 0$, see for example [For91]. For the basic results in logarithmic geometry we refer to [Lit82].

Now, the positivity of the logarithmic canonical line bundle is not unexpected. In fact, given a logarithmic pair (X, D) with D a simple normal crossing divisor, the existence of a Kähler-Einstein metric with negative scalar curvature on $X \setminus D$ and Poincaré singularities along D is guaranteed under the condition that $K_X + D$ is an ample divisor; see the works of R. Kobayashi [Kob84] and Cheng-Yau [CY86]. Let us briefly discuss the proof of such results. This existence problem reduces to the study of a complex Monge-Ampère equation of the form

$$(2) \quad (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^\varphi \Psi$$

where

$$\Psi = \frac{\Omega}{\prod_i \|s_i\|^2 (\log \|s_i\|^2)^2}$$

for some globally defined volume form Ω on X and where ω is a Carlson-Griffiths [CG72] Kähler metric on $X \setminus D$. More precisely, ω is a complete Kähler metric on $X \setminus D$ with Poincaré singularities along D which is obtained as minus the Ricci curvature of Ψ . As explained in [CG72], if we assume the log-canonical divisor to be ample, this is always possible by appropriately choosing Ω and the Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}_X(D_i)$. Now, as proved in [Kob84] and [CY86] the solution ω_φ of (2) has Poincaré type singularities. Thus, the Poincaré-Lelong formula [GH78] gives the following.

Remark 2.5. *The Kähler current ω_φ solving (2) satisfies a distributional equation of the form*

$$\text{Ric}_{\omega_\varphi} - \sum_i [D_i] = -\omega_\varphi,$$

where by $[D]$ we indicate the current of integration along D .

It is easy to construct Carlson-Griffiths type metrics under slightly different positivity conditions. For example, let us consider a collection of positive numbers $\alpha_i \in (0, 1]$ and assume the *twisted* log-canonical divisor

$$K_X + \sum_i \alpha_i D_i$$

to be ample on X . Then, a simple modification of the original argument given in [CG72] can be used to construct a Kähler metric on $X \setminus D$ with Poincaré singularities along D which is in the cohomology class of the \mathbb{R} -divisor $K_X + \sum_i \alpha_i D_i$. As in the untwisted case, one can then try to deform this background metric to a complete Kähler-Einstein metric. Nevertheless, this approach seems to not work in this context. In fact, Kähler-Einstein metrics with Poincaré type singularities are complete and therefore quite rigid because of Yau's generalized Schwartz lemma.

This is why, in the definition below, we limit ourselves to the case where all the α_i 's are identically equal to one. Compare with Definition 2.9 below.

Definition 2.6. A Kähler current $\hat{\omega}$ with Poincaré singularities along the D_i 's is called Kähler-Einstein if it satisfies a distributional equation of the form

$$(3) \quad Ric_{\hat{\omega}} - \sum_i [D_i] = -\hat{\omega}$$

where by $[D]$ we indicate the current of integration along D .

We can now derive the necessary condition for the existence of a Kähler-Einstein metric with Poincaré singularities.

Proposition 2.7. Let $\hat{\omega}$ be a Kähler-Einstein metric with Poincaré singularities as in 3, then the divisor $K_X + \sum_i D_i$ is ample.

Proof. Since by assumption $\hat{\omega}$ satisfies 3, we have that the cohomology class of the divisor $K_X + \sum_i D_i$ can be represented by a Kähler current. By the structure of the pseudo-effective cone given in [Dem92], we conclude that $K_X + \sum_i D_i$ is a big divisor. Next, we want to show that $K_X + \sum_i D_i$ has to be an ample divisor. First, a simple computation shows that the *Lelong* numbers of $\hat{\omega}$ are zero at any smooth point of D . Then a regularization argument based on results of Demailly [Dem92] can be used to show that $K_X + \sum_i D_i$ is indeed an ample divisor. \square

We now briefly discuss the existence problem for incomplete Kähler-Einstein metrics on quasi-projective varieties. The most natural kind of incomplete metrics appearing in Kähler geometry seem to have *cone-edge* singularities. The study of such metrics was originally proposed by G. Tian in [Tia96]. Thus, given a pair (X, D) as above, let us recall the notion of a Kähler metric with edge singularities along D . Let $\{D_i\}$ be the irreducible smooth components of D and consider a collection of positive numbers $\alpha_i \in (0, 1)$. A smooth Kähler metric $\hat{\omega}$ on $X \setminus D$ is said to have edge singularities of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's if for any point $p \in D$ and coordinate neighborhood $(\Omega; z_1, \dots, z_n)$ centered at p for which

$$D|_{\Omega} = \{z_1 \cdot \dots \cdot z_k = 0\},$$

then $\hat{\omega}$ is *quasi-isometric* in Ω to the following model metric

$$(4) \quad \omega_0 = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2\alpha_i}} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right).$$

It is easy to construct Kähler metrics with edge singularity along a divisor.

Example 2.8. Let ω be a Kähler metric on X . For simplicity let D be a smooth reduced divisor and choose a real number $\alpha \in (0, 1)$. Choose an Hermitian metric $\|\cdot\|$ on $\mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{O}(D))$ be a defining section for the divisor D . Then for $T > 0$ big enough

$$\hat{\omega} = T\omega + \sqrt{-1}\partial\bar{\partial}\|s\|^{2(1-\alpha)}$$

defines Kähler metric on $X \setminus D$ with edge singularities of cone angle $2\pi(1 - \alpha)$ along D .

As first observed in [Jef96], $\hat{\omega}$ is indeed an incomplete Kähler metric of finite volume on $X \setminus D$. Now, because of the finite volume property, standard results in the theory of currents [Dem01] imply that $\hat{\omega}$ can be regarded as a *Kähler current* on X with *singular support* D . The fact that $\hat{\omega}$ defines a Kähler current and not only a closed positive current follows from the quasi-isometric condition given in 4.

We can now introduce the definition of a Kähler-Einstein metric with edge singularities.

Definition 2.9 (Tian [Tia96]). *A Kähler current $\hat{\omega}$ with edges of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's is called Kähler-Einstein with curvature λ if it satisfies the distributional equation*

$$(5) \quad Ric_{\hat{\omega}} - \sum_i \alpha_i [D_i] = \lambda \hat{\omega}$$

where by $[D]$ we indicate the current of integration along D .

A simple regularization argument based on results of Demailly [Dem92] can now be used to derive the *necessary* condition for the existence of a Kähler-Einstein metric with edge singularities.

Proposition 2.10. *Let $\hat{\omega}$ be a Kähler-Einstein metric with edge singularities as in 5 with $\lambda = -1$ ($\lambda = 1$), then the \mathbb{R} -divisor $K_X + \sum_i \alpha_i D_i$ ($-K_X - \sum_i \alpha_i D_i$) is ample.*

Proof. See Propositions 2.1 and 2.2 in [DiC12b]. □

To sum up, the entire set of questions concerning the existence and limiting behavior of families of Kähler-Einstein metrics on quasi-projective varieties requires a precise understanding of the positivity properties of certain log-canonical bundles. This is the object of study in Sections 4-6. Since we are primarily interested in taking limits as the cone angles go to 0 or 2π we limit ourselves to the case where all the α_i 's are identical.

3. SOME THEOREMS FROM ALGEBRAIC GEOMETRY

In this section we recall some fundamental results in algebraic geometry. We follow the notation and terminology of [KM98]. However we state here some definitions we will need later.

Definition 3.1. *A 1-cycle is a formal linear combination of irreducible, reduced and proper curves $C = \sum a_i C_i$. Two 1-cycles C, C' are called numerically equivalent if $(C \cdot D) = (C' \cdot D)$ for any Cartier divisor D . 1-cycles with real coefficients modulo numerical equivalence form a \mathbb{R} -vector space and we denote it by $N_1(X)$.*

We are interested in particular subspaces of $N_1(X)$.

Definition 3.2. *Let X be a proper variety.*

$$NE(X) := \left\{ \sum a_i [C_i] \mid C_i \subset X, 0 \leq a_i \in \mathbb{R} \right\} \subset N_1(X).$$

Let $\overline{NE}(X)$ be the closure of $NE(X)$ in $N_1(X)$.

For any divisor D , set $D_{\geq 0} := \{x \in N_1(X) \mid D \cdot x \geq 0\}$ and similarly for > 0 , ≤ 0 and < 0 . Finally

$$\overline{NE}(X)_{D \geq 0} := \overline{NE}(X) \cap D_{\geq 0}.$$

In birational geometry and especially in the minimal model program, we need to deal with some classes of singularities of pairs (X, D) . The definitions are a little bit technical but the idea is simple. For a singular variety X we want to measure its singularities comparing K_X and f^*K_X , where f is a resolution of the singularities of X . In case we have a pair (X, D) we will measure the singularities of X and D together.

Definition 3.3. A pair (X, D) consists of a normal variety X and a \mathbb{R} -Weil divisor D such that $K_X + D$ is \mathbb{R} -Cartier.

Let (X, D) be a pair. A log resolution of the pair is a birational map $f : Y \rightarrow X$ such that Y is smooth, the exceptional set $Ex(f)$ of f is a divisor and $Ex(f) \cup f^{-1}(\text{Supp } D)$ is a simple normal crossing divisor. For any log resolution of (X, D) we can write $K_Y + \Gamma \sim_{\mathbb{Q}} f^*(K_X + D)$, where $\Gamma = \sum a_i \Gamma_i$ and Γ_i are distinct reduced irreducible divisors. $1 - a_i$ is called the log discrepancy of Γ_i with respect to (X, D) .

Definition 3.4. Let (X, D) be a pair. We say that (X, D) is a klt (resp. lc) pair if there exists a log resolution $f : Y \rightarrow X$ as above such that $a_i < 1$ (resp. $a_i \leq 1$).

The abbreviation klt stands for Kawamata log terminal and lc for log canonical. In this paper we will mainly deal with pairs (X, D) where X is a smooth variety. In this case the discrepancies measure how singular is the effective divisor D . In case D has simple normal crossing support then we get only a restriction on the coefficients of D . More precisely we have the following result, see Corollary 3.12 in [Kol97] for a proof.

Proposition 3.5. Let X be a smooth variety and $D = \sum d_i D_i$ an effective divisor with simple normal crossing support. Then (X, D) is klt (resp. lc) if and only if $d_i < 1$ (resp. $d_i \leq 1$) for all i .

Roughly speaking the goal of the minimal model program is to find the simplest variety birationally equivalent to a given variety. The natural way to measure how simple a variety can be, is to look at its canonical divisor and its intersection with curves on the variety. So it is fundamental to understand $(K_X)_{\leq 0}$ inside $\overline{NE}(X)$. The celebrated Cone Theorem describe how the negative part looks like: it is generated by countably many rational curves with bounded intersection with K_X . For details see [KM98].

Theorem 3.6 (Cone theorem). Let (X, D) be a lc pair. Then there are countably many rational curves $C_j \subset X$ such that $0 < -(K_X + D) \cdot C_j \leq 2n$ and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + D) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

If X is smooth and $D = \emptyset$ then we can choose C_j such that $0 < -K_X \cdot C_j \leq n + 1$.

In [KM98] the cone theorem is proved for klt pairs. The case of lc pairs is treated in [Amb03] and [Fuj09].

It will be important for us to understand if certain adjoint linear systems are semi-ample. Our main tool to study semi-ampleness is the base point free theorem. It is one of the main step in the proof of the cone theorem and it is of great interest on its own. Recall that a divisor L is big if it has maximal Kodaira dimension and it is nef if $L \cdot C \geq 0$ for any irreducible curve C .

Theorem 3.7 (Base point free theorem). *Let (X, D) a klt pair and let L be a nef Cartier divisor such that $aL - (K_X + D)$ is big and nef for some $a > 0$. Then the linear system $|mL|$ is base point free for any $m \gg 0$.*

See [KM98] for the proof of the base point free theorem and more details.

A divisor is said to be strictly nef if $L \cdot C > 0$ for any irreducible curve C . Note that a strictly nef divisor is not necessarily ample even if it is big, see Section 5 and [Har70] for examples. On the other hand the base point free theorem implies the following.

Corollary 3.8. *Let (X, D) be a klt pair such that $K_X + D$ is big and strictly nef. Then $K_X + D$ is ample.*

Proof. Since $K_X + D$ is big and nef by Theorem 3.7 it is semi-ample. Then for m big enough $m(K_X + D)$ is base point free and it defines a morphism $\phi_{|m(K_X + D)|} : X \rightarrow Y \subseteq \mathbb{P}^k$, where $k = \dim H^0(X, \mathcal{O}_X(m(K_X + D))) - 1$. Suppose C is a curve in some fiber of ϕ then $(K_X + D) \cdot C = 0$, but this is impossible since $K_X + D$ is strictly nef. Then ϕ is a finite morphism and Corollary 1.2.15 in [Laz04a] implies that $K_X + D$ is ample. \square

Note that the same proof tells us that strictly nef and semi-ample divisors are ample.

The base point free theorem is one of the first step toward a solution of the following conjecture.

Conjecture 3.9 (Abundance). *Let (X, D) a lc pair such that $K_X + D$ is nef. Then $K_X + D$ is semi-ample.*

The abundance conjecture in such generality it is known only if $\dim(X) \leq 3$, see [Kol92], [KMM94] and [KMM04]. Furthermore the base point free theorem implies that if (X, D) is a klt pair with $K_X + D$ big and nef then $K_X + D$ is semi-ample.

Note that Conjecture 3.9 is not a trivial statement even for surfaces. If the abundance conjecture holds then it implies that if $K_X + D$ is strictly nef then it is ample.

The difference between abundance for klt pairs and abundance for lc pairs is very subtle. The base point free theorem, as stated in Theorem 3.7, it is not know for lc pairs. Actually Theorem 3.7 for lc pairs in dimension $n + 1$ implies the abundance conjecture for klt pairs of any Kodaira dimension in dimension n . In particular it implies abundance with X smooth and $D = \emptyset$. See Theorem A.6 in [Laz09] for details.

In the study of Kähler-Einstein metrics is fundamental to understand where the map associated to some power of a big and nef line bundle defines an embedding into some projective space. This locus can be understood using a refinement of the stable base locus. We recall here the key definition and we refer to Definition 10.3.2 in [Laz04b] and the discussion there for more details.

Definition 3.10. *The stable base locus of a divisor L is*

$$\mathbf{B}(L) := \bigcap_{m \geq 1} \text{Bs}(|mL|),$$

where $\text{Bs}(|L|)$ is the base locus of L .

The augmented base locus of a divisor L is the Zariski-closed set

$$\mathbf{B}_+(L) := \mathbf{B}(L - \epsilon A),$$

for any ample A and sufficiently small $\epsilon > 0$.

Note that $\mathbf{B}_+(L)$ is a closed subset of X if and only if L is big and $\mathbf{B}_+(L) = \emptyset$ if and only if L is ample. In Section 5 we will see how $\mathbf{B}_+(L)$ is related to the locus where the map $\phi_{|mL|}$ is not an embedding, for m sufficiently large.

Quite remarkably the augmented base locus of a big and nef divisor can be described numerically thanks to a theorem of Nakamaye. See Theorem 10.3.5 in [Laz04b].

Theorem 3.11. *Let X be a smooth projective variety and let L be a big and nef divisor on X . Then $\mathbf{B}_+(L)$ is the union of all positive dimensional subvarieties $V \subseteq X$ such that $L^{\dim(V)} \cdot V = 0$.*

Note that if a subvariety $V \subseteq X$ is such that $L^{\dim(V)} \cdot V = 0$ then $V \subseteq \mathbf{B}_+(L)$, so the content of the theorem is to prove the converse.

4. POSITIVITY FOR VARIETIES OF GENERAL AND LOG-GENERAL TYPE

As recalled in Section 2, given a pair (X, D) for which $K_X + D$ is ample then there exists a Kähler-Einstein metric on $X \setminus D$ with Poincaré singularities along D . Following intuitions of G. Tian [Tia96], this Kähler-Einstein metric should arise as the limit of a 1-parameter family of Kähler-Einstein metrics with cone-edge singularities along D . It would then be desirable to have a uniform interval of angles, depending on the dimension only, for which the existence of such metrics is unobstructed, see Proposition 2.10. Thus, to warm up, we study the following.

Question 4.1. *Let X^n be a smooth variety and let D be a reduced effective divisor such that $K_X + D$ is ample. Is there a fixed $\alpha_n < 1$, which depends only on n , such that $K_X + \alpha D$ is ample for any $\alpha \in (\alpha_n, 1]$?*

In order to deal with the previous problem we need to study the nef threshold.

Definition 4.2. *Let X be a smooth projective variety and let D be an integral divisor such that $K_X + tD$ is nef for some $t \in \mathbb{R}_{\geq 0}$. The nef threshold is*

$$r(X, D) := \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is nef}\}.$$

Since ampleness is an open condition, see Proposition 1.3.7 in [Laz04a], we know that $r(X, D) < 1$ in Question 4.1. The content of the question is then if it is possible to bound the nef thresholds away from one uniformly for all varieties of fixed dimension. In order to obtain a uniform bound we use the following well known corollary of the cone theorem.

Lemma 4.3. *Let X be a smooth projective variety with $\dim(X) = n$ and let H be an ample divisor. Then $K_X + (n+1)H$ is nef.*

Proof. By Theorem 3.6 we know that any curve C in X is numerically equivalent to

$$a_1 C_1 + \cdots + a_r C_r + F,$$

where a_i are positive real numbers, C_i are rational curves such that $0 > C_i \cdot K_X \geq -(n+1)$ and $K_X \cdot F \geq 0$. Since H is ample, we have $H \cdot C_i \geq 1$. Then $(K_X + (n+1)H) \cdot C \geq 0$. \square

The proposition below gives the desired uniform bound for the nef threshold when $K_X + D$ is ample.

Proposition 4.4. *Let X be a smooth projective variety of dimension n and let D be a divisor such that $K_X + D$ is ample. Then*

$$r(X, D) \leq \frac{n+1}{n+2}.$$

Proof. Since $K_X + D$ is ample by Lemma 4.3 we know that $K_X + (n+1)(K_X + D)$ is nef. In particular $(n+2)K_X + (n+1)D$ is nef and then the result follows. \square

Let us observe that the bound given in Proposition 4.4 is *sharp*.

Remark 4.5. *Take $X = \mathbb{P}^n$ and $D = (n+2)H$ where H is the hyperplane section. Then $K_{\mathbb{P}^n} + D$ is ample and $r(\mathbb{P}^n, D) = \frac{n+1}{n+2}$.*

The following corollary provides a satisfactory answer to Question 4.1.

Corollary 4.6. *Let X and D as in the Proposition 4.4. Then for any $\alpha \in (\frac{n+1}{n+2}, 1]$ the divisor $K_X + \alpha D$ is ample.*

Proof. The result follows from the fact that in this case the nef threshold can be obtained as $r(X, D) = \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is ample}\}$. \square

In the above discussion, we restricted ourselves to pairs (X, D) with $K_X + D$ ample. Nevertheless, it is easy to construct examples of pairs such that $X \setminus D$ admit complete negatively curved Kähler-Einstein metrics of finite volume, but for which $K_X + D$ is *not* ample. A wealth of examples is provided by the theory of locally symmetric varieties and their compactifications, see for example [AMRT10]. More concretely, let us consider the following class of low-dimensional examples. For more details the interested reader may also refer to Section 4 in [DiC12a].

Example 4.7. *Let X^2 be a smooth toroidal compactification of a finite-volume complex-hyperbolic surface and let D be the compactifying divisor. The set D simply consists of smooth disjoint elliptic curves. Note that $X \setminus D$ admits a natural*

finite-volume Kähler-Einstein metric coming from the Bergman metric on \mathbb{CH}^2 . Nevertheless, the line bundle $K_X + D$ is just big and nef since $(K_X + D) \cdot D = 0$.

The particular class of toroidal compactifications considered in Example 4.7 has other remarkable properties. For a proof of the following fact see Theorem 3.4 in [DiC12b].

Fact 4.8. *Let (X, D) be as in Example 4.7. Then for any α close enough the divisor $K_X + \alpha D$ is ample.*

It is therefore natural to address the following set of problems.

Question 4.9. *Let X^n be a smooth variety and let D be a reduced effective divisor such that $K_X + D$ is big and nef. Do we always have $r(X, D) < 1$? Moreover, if $r(X, D) < 1$, can we find $\alpha_n < 1$ depending only on n such that $K_X + \alpha D$ is big and nef for any $\alpha \in (\alpha_n, 1]$? Finally, can we characterize the pairs (X, D) for which $K_X + \alpha D$ is ample for α close enough to one?*

When dealing with these kind of questions is probably best to start analyzing a weak notion of positivity: bigness. Thus, analogously to the nef threshold, we define the pseudo-effective threshold.

Definition 4.10. *Let X be a smooth projective variety and let D be an integral divisor such that $K_X + tD$ is big for some $t \in \mathbb{R}_{\geq 0}$. The pseudo-effective threshold is*

$$\tau(X, D) := \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is pseudo-effective}\}.$$

Recall that the big cone is open and its closure is the cone of pseudo-effective divisors, see Theorem 2.2.26 in [Laz04a]. Then if $K_X + D$ is big we immediately obtain that $\tau(X, D) < 1$. Since nef divisors are also pseudo-effective we have that $\tau(X, D) \leq r(X, D)$. Let us construct an example for which these numerical invariants are actually different, compare with Question 4.9.

Example 4.11. *Let X a smooth projective surface and D a smooth irreducible divisor such that $K_X + D$ is ample. Let $\pi : Y \rightarrow X$ be the blow up of a point in D . Let D' be the strict transform of D on Y . Then $K_Y + D'$ is big and nef but $K_Y + \alpha D'$ is not nef for any $\alpha < 1$. Then $r(Y, D') = 1$ but $\tau(Y, D') < 1$.*

The phenomena explained in Example 4.11 relies on the fact that, differently from the cone of big divisor, the cone of nef divisors is *closed* being the closure of the ample cone, see Theorem 1.4.23 in [Laz04a]. Thus, we do not expect the nefness to hold when we decrease the coefficients of D . Remarkably, the bigness of $K_Y + D'$ in Example 4.11 does not help in achieving $r(Y, D') < 1$.

We are now interested in trying to characterize the pairs (X, D) with $K_X + D$ nef and with nef threshold strictly less than one. The following result gives a quite satisfactory answer, compare with Question 4.9.

Proposition 4.12. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then $r(X, D) < 1$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$.*

Proof. Suppose $r(X, D) < 1$. Let C be an irreducible curve such $(K_X + D) \cdot C = 0$. In particular we have that $D \cdot C = -K_X \cdot C$. We want to prove that $K_X \cdot C \geq 0$. Let α be close to one such that $K_X + \alpha D$ is nef. Then

$$(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C \geq 0.$$

Conversely suppose that there are no irreducible curves C with the properties $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. We would like to show that $r(X, D) < 1$. Let C' be an irreducible curve such that $(K_X + D) \cdot C' > 0$. By the same argument in the proof of Lemma 4.3 we can write $C' \equiv a_1 C_1 + \cdots + a_r C_r + F$ where a_i are positive numbers, C_i are irreducible curves with $0 > K_X \cdot C_i \geq -(n+1)$ and $K_X \cdot F \geq 0$. Since F is a limit of effective 1-cycles and $K_X + D$ is nef we have that $(K_X + D) \cdot F \geq 0$. By assumption if $(K_X + D) \cdot C_i = 0$ then $K_X \cdot C_i \geq 0$. In particular $(K_X + (n+1)(K_X + D)) \cdot C' \geq 0$. This implies that $(K_X + \frac{n+1}{n+2}D) \cdot C' \geq 0$ for any curve C' such that $(K_X + D) \cdot C' > 0$. Then we need only to check the nefness of $K_X + \alpha D$ on irreducible curve C such that $(K_X + D) \cdot C = 0$. The same computation as before tells us that $(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C \geq 0$. \square

Let us observe that Proposition 4.12 implies the following.

Corollary 4.13. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then either $r(X, D) = 1$ or $r(X, D) \leq \frac{n+1}{n+2}$.*

The reasoning given in Proposition 4.12 is easily adapted to understand the obstructions for $K_X + \alpha D$ to be *strictly* nef. In fact, we can state:

Proposition 4.14. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then $K_X + \alpha D$ is strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

We are now ready to address the last problem considered in Question 4.9.

Theorem 4.15. *Let X be a smooth projective variety and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Then $K_X + \alpha D$ is ample for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

Proof. Suppose $K_X + \alpha D$ to be ample. Let C be an irreducible curve such that $(K_X + D) \cdot C = 0$. Then

$$(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C > 0.$$

Conversely, by Proposition 4.14 we know that $K_X + \alpha D$ is strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$. Since $\tau(X, D) \leq r(X, D)$ we conclude that $K_X + \alpha D$ is big and strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$. Note that $(X, \alpha D)$ is a klt pair. By Theorem 3.7 we know that $K_X + \alpha D$ is semi-ample. Since strictly nef semi-ample divisors are ample the proof is complete. \square

We can now give a proof of Theorem 1.1 stated in the Introduction 1.

Proof of Theorem 1.1. Combine Proposition 2.10 and Theorem 4.15 with the analytical results of Campana-Guenancia-Păun [CGP11] and Mazzeo-Rubinstein [MR12], more precisely Theorem A in [CGP11] and Theorem 1.3 in [MR12]. \square

Theorem 4.15 nicely applies in low-dimensions. Thus, let us collect few corollaries concerning complex surfaces.

Corollary 4.16. *Let X be a smooth projective surface and let D be a divisor such that $K_X + D$ is big and nef. Suppose there are no curves C in X such that $C = \mathbb{P}^1$, $C^2 = -1$ and $C \cdot D = 1$, then $r(X, D) < 1$.*

Proof. We want to apply Proposition 4.12. Let C be a curve such that $(K_X + D) \cdot C = 0$. Then by the Hodge index theorem $C^2 < 0$. By adjunction we have

$$K_X \cdot C = 2g - 2 - C^2.$$

Suppose $K_X \cdot C < 0$. Since $C^2 < 0$ this can happen if and only if $g = 0$ and $C^2 = -1$. But then $D \cdot C = -K_X \cdot C = 1$ which contradicts our hypothesis. \square

Similarly, we have a nice characterization of the pairs with ample twisted log-canonical bundles.

Corollary 4.17. *Let X be a smooth projective surface and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Suppose there are no curves C in X such that $C = \mathbb{P}^1$, $C^2 = -1$ and $C \cdot D = 1$, or $C = \mathbb{P}^1$, $C^2 = -2$ and $C \cdot D = 0$. Then $K_X + \alpha D$ is ample for any $\alpha \in (\frac{3}{4}, 1)$.*

Proof. We want to apply Theorem 4.15. Because of Corollary 4.16, it suffices to characterize the curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C = 0$. By the Hodge index theorem we have $C^2 < 0$ and then by adjunction $K_X \cdot C = 0$ if and only if $g = 0$ and $C^2 = -2$. This implies $C \cdot D = 0$. \square

The interested reader should compare Corollaries 4.16 and 4.17 with Theorems 3.3 and 3.4 in [DiC12b]. The results presented in [DiC12b] contain a geometric characterization of the pairs (X^2, D) with $K_X + D$ big and nef. Such results rely on F. Sakai's notions of semi-stability and D -minimality, see [Sak80].

It is now interesting to collect a corollary regarding Fano varieties. Recall that X is called a Fano variety if $-K_X$ is ample.

Corollary 4.18. *Let X be a smooth Fano variety. Let D be a divisor such that $K_X + D$ is nef. Then $r(X, D) < 1$ if and only if $K_X + D$ is strictly nef.*

Proof. If $K_X + D$ is strictly nef the result follows from the same argument of Lemma 4.3. Assume $r(X, D) < 1$. Since X is Fano we have that $-K_X$ is ample and in particular $K_X \cdot C < 0$ for any irreducible curve C . Then by Proposition 4.12 we must have $(K_X + D) \cdot C > 0$ for any irreducible curve C , i.e. $K_X + D$ is strictly nef. \square

Corollary 4.19. *Let X be a smooth Fano variety with $\dim(X) \leq 3$. Let D be an effective divisor such that (X, D) is lc and $K_X + D$ is nef. Then $r(X, D) < 1$ if and only if $K_X + D$ is ample.*

Proof. By the above corollary $r(X, D) < 1$ if and only if $K_X + D$ is strictly nef. Since $\dim(X) \leq 3$ the abundance conjecture holds and then $K_X + D$ is semi-ample. The result follows from the fact that strictly nef and semi-ample divisors are ample. \square

Of course the above corollary should be true in any dimension since it follows from the abundance conjecture. In the mean time, waiting for abundance to be proved, we derive a corollary for *toric* Fano varieties.

Corollary 4.20. *Let X be a smooth toric Fano variety. Let D be an effective divisor such that (X, D) is lc and $K_X + D$ is nef. Then $r(X, D) < 1$ if and only if $K_X + D$ is ample.*

Proof. On a smooth toric variety every nef divisor is globally generated, see Theorem 6.3.12 in [CLS11]. \square

In the discussion above, we mainly focused on the study of the nef threshold. We now would like to better understand the behavior of the pseudo-effective threshold. First, if $r(X, D) < 1$ then $\tau(X, D)$ is automatically bounded away from one. It remains to understand if the same is true in the case $r(X, D) = 1$. The fact that $K_X + D$ is assumed to be nef turns out to be very useful. In fact, recall the following theorem of Andreatta, see Theorem 5.1 in [And11].

Theorem 4.21 (Andreatta). *Let X be a smooth projective variety of dimension n and let L be a big and nef divisor. Then $K_X + (n + 1)L$ is pseudo-effective.*

It is interesting to notice how Theorem 4.21 plays a similar role as Lemma 4.3. Thus, the idea of the proof of Proposition 4.4 gives the following.

Proposition 4.22. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is big and nef. Then*

$$\tau(X, D) \leq \frac{n+1}{n+2}.$$

As expected, we have:

Corollary 4.23. *Let X and D as in the Proposition 4.22. Then for any $\alpha > \frac{n+1}{n+2}$ the divisor $K_X + \alpha D$ is big.*

Let us now briefly discuss another approach to the problem addressed in Proposition 4.22. Recall the following result of Kollár [Kol97].

Theorem 4.24 (Kollár). *Let X be a smooth variety and let L be a big and nef divisor. Then $K_X + mL$ gives a birational map for any $m \geq \binom{n+2}{2}$.*

Note that, given L as in 4.24, then $K_X + \binom{n+2}{2}L$ is not only pseudo-effective but also big. Thus, by letting $L = K_X + D$ and proceeding as in Proposition 4.22 we obtain $\tau(X, D) \leq \frac{\binom{n+2}{2}}{1 + \binom{n+2}{2}}$. Despite the fact that this new bound is worse than the one given in Proposition 4.22, this new approach, based on Theorem 4.24, provides the idea of how to extend the result when $K_X + D$ is only big. In fact, it strongly suggests that results in effective birationality can help in bounding the pseudo-effective threshold. Recently Hacon, McKernan and Xu generalized Theorem 4.24 in [HMX12] and they proved the following.

Theorem 4.25 (Hacon-McKernan-Xu). *Let X be a smooth projective variety and let D be a reduced effective divisor with simple normal crossing such that $K_X + D$ is big. Then there exists a positive number α_n depending only on the dimension of X such that $K_X + \alpha D$ is big for all $\alpha \in (\alpha_n, 1]$.*

Actually they proved a more general statement when (X, D) is a lc pair and the coefficients of D are in a fixed DCC set I .

In the rest of the section we address the following:

Question 4.26. *Let X be a smooth variety with K_X big and nef and let D be an effective divisor. Can we characterize the pairs (X, D) with $K_X + \alpha D$ big and nef or ample for α close enough to zero? Finally, is there a fixed $\alpha_n > 0$, which depends only on n , such that $K_X + \alpha D$ is big and nef or ample for any $\alpha \in [0, \alpha_n)$?*

Note that in this case $K_X + \alpha D$ is big for any $\alpha \geq 0$, so the main problem is to understand when it is nef. To this aim we use again the cone theorem, but now we need (X, D) to be a lc pair. Thus, to warm up, let us observe the following.

Proposition 4.27. *Let (X, D) be a lc pair with K_X ample. Then $K_X + \alpha D$ is ample for any*

$$\alpha < \frac{1}{2n+1}.$$

Proof. By the cone theorem for lc pairs $K_X + D + 2nK_X$ is nef. In particular so is $K_X + \frac{1}{2n+1}D$. \square

A similar approach can now be applied when K_X is simply nef.

Theorem 4.28. *Let X be a smooth projective variety with K_X nef. Let D be a reduced effective divisor with simple normal crossing support. Then $K_X + \alpha D$ is nef for $\alpha \in [0, \frac{1}{2n+1}]$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C < 0$ and $K_X \cdot C = 0$.*

Proof. By the cone theorem for lc pairs, if there are no irreducible curves C such that $(K_X + D) \cdot C < 0$ and $K_X \cdot C = 0$ then the \mathbb{R} -divisor

$$K_X + D + tK_X$$

is nef for $t \geq 2n$. Conversely, if $K_X + \alpha D$ is nef then for some $\alpha \in (0, 1)$ then the curves that dot negatively with $K_X + D$ must intersect positively with K_X . \square

We conclude this section by studying when $K_X + \alpha D$ is ample for α close to zero.

Theorem 4.29. *Let X be a smooth projective variety with K_X big and nef. Let D be a reduced effective divisor with simple normal crossing support. Then $K_X + \alpha D$ is ample for $\alpha \in (0, \frac{1}{2n+1})$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$.*

Proof. If there are no irreducible curves C such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$ we have that $K_X + \alpha D$ is strictly nef for any $\alpha \in (0, \frac{1}{2n+1})$. Recall that $K_X + \alpha D$ is a big \mathbb{R} -divisor for any $\alpha \geq 0$. Then Theorem 3.7 implies that $K_X + \alpha D$ is semi-ample and therefore ample. The converse should be clear by now. \square

We do not expect the above result to be sharp.

We conclude this section by giving the proof of Theorem 1.2 stated in the Introduction 1.

Proof of Theorem 1.2. Combine Proposition 2.10 and Theorem 4.29 with Theorem 1.3 in [MR12]. \square

5. REMARKS ON AMPLENESS FOR LINE BUNDLES ON QUASI-PROJECTIVE VARIETY

It is clear that the study of Kähler-Einstein metrics on quasi-projective varieties must be connected with the notion of *ampleness* on open algebraic manifolds. It seems there is no general agreement in the existing literature on the definition of this concept. Recall that for compact manifold the notion of ampleness is usually introduced algebraically and then formulated in geometric and numerical terms, see Chapter 1 in [Laz04a]. Here, we follow a completely analogous path. The interested reader should compare the results contained in this section with some of the existing literature [Wu08], [Wu09], [TY87].

Thus, let us start with a definition. For more details see [Tak93].

Definition 5.1. *Let X be a smooth projective variety and let D be a simple normal crossing divisor. A line bundle L on X is said to be ample modulo D , if for any torsion free sheaf \mathcal{F} on X , there exists an integer m_0 such that $(\mathcal{F} \otimes L^n)_{|X \setminus D}$ is generated by $H^0(X, \mathcal{F} \otimes L^n)$ for $m \geq m_0$.*

We then say that a divisor L is very ample modulo D if the rational map $\phi_{|L|}$ associated to the linear system $|L|$ defines an embedding of $X \setminus D$ into some projective space. By the same argument of Theorem II.7.6 in [Har77], one can show that some power of an ample modulo D divisor is very ample modulo D . In particular it follows that if L is ample modulo D then it is necessarily big. Let us continue with some general observations. First, note that $\mathbf{B}_+(L)$ is a proper subset of X since L is big. Next, recall that $\mathbf{B}_+(L)$ can be written as

$$\mathbf{B}_+(L) = \bigcap_{L=A+E} \text{Supp}(E),$$

where the intersection is taken over all decompositions $L = A + E$, where A is ample and E effective, see Remark 1.3 in [ELMNP06]. Since X is noetherian and $\mathbf{B}_+(L)$ is a Zariski closed subset of X we can find finitely many decompositions $L = A_i + E_i$ such that $\mathbf{B}_+(L) = \bigcap_{i=1}^k \text{Supp}(E_i)$. In particular this implies that for m large enough, the linear system $|mL|$ defines an embedding of $X \setminus \mathbf{B}_+(L)$ into some projective space. This simple argument gives the following.

Fact 5.2. *If L is ample modulo D then $\mathbf{B}_+(L) \subseteq D$.*

It would now be desirable to have a numerical characterization of ampleness modulo D .

Definition 5.3. *Let X be a smooth projective variety and let D be an effective divisor. A line bundle L on X is said to be strictly nef modulo D , if $L \cdot C > 0$ for any curve C not entirely contained in D .*

Let us observe that, similarly to the compact case, being strictly nef modulo D is not enough for achieving ampleness modulo D . To this aim recall that a strictly nef divisor is not necessarily ample. The first construction of such a divisor is due to Mumford. In fact, he showed the existence of a surface X with a strictly nef divisor D such that $D^2 = 0$. Ramanujam, based on Mumford's example, showed the existence of a big and strictly nef divisor that is not ample. Note that by Nakai's criterion a divisor which is big and strictly nef but not ample can exist in dimension greater or equal than three only. For the convenience of the reader we sketch the construction of these examples. For details see Chapter I §10 in [Har70].

Example 5.4 (Mumford, Ramanujam). *We start with Mumford's example. Let C be a smooth curve of genus $g \geq 2$. By a theorem of Seshadri there exists a stable vector bundle E of degree zero over C of rank two such that all its symmetric powers are stable. Let $X = \mathbb{P}(E)$ and $D = \mathcal{O}_X(1)$ be the Serre line bundle. One can check that D is strictly nef. Since E has degree zero, $D^2 = 0$ and in particular D is not ample. Now we can give Ramanujam's example of a big and strictly nef divisor which is not ample. We will show the existence of such divisor using Mumford's example. Let X be as above and let H be an effective ample divisor on X . Define $Y := \mathbb{P}(\mathcal{O}_X(D - H) \oplus \mathcal{O}_X)$ and let $\pi : Y \rightarrow X$ be the projection. Let $X_0 \subseteq Y$ be the zero section. Finally let $L := X_0 + \pi^*H$. As shown in [Har70], L is strictly nef but it is not ample because $L^2 \cdot X_0 = D^2 = 0$. Furthermore L is big because $L^3 = (D + H) \cdot H > 0$.*

Thus, let L be a big and strictly nef divisor which is not ample. Observe that $\mathbf{B}_+(L)$ is not empty otherwise L would be ample. Then L is strictly nef modulo D for any effective divisor D . Nevertheless, by choosing a divisor D that does not contain $\mathbf{B}_+(L)$ we certainly obtain that L cannot be ample modulo D , see Fact 5.2.

The discussion above tells us that if we want to control $\mathbf{B}_+(L)$ only with intersection numbers we have to take in account all positive dimensional subvarieties and not only curves. In fact, let L be nef and such that $L^k \cdot V > 0$ for any k -dimensional subvariety not entirely contained in D . Then by Theorem 3.11 we have that $\mathbf{B}_+(L)$ must be contained in D . This is then enough to conclude:

Fact 5.5. *Let L be nef such that $L^k \cdot V > 0$ for any k -dimensional subvarieties not entirely contained in D . Then L is ample modulo D .*

Let us now study line bundles which are semi-ample. We would like to show that if L is semi-ample and strictly nef modulo D then it must be ample modulo D . First, since L is semi-ample then there exists a positive integer m such that the linear system $|mL|$ defines a morphism into some projective space. Let us

denote this map by $\phi_{|mL|}$. Since L is strictly nef modulo D we have that $\phi_{|mL|}$ is a finite map over $X \setminus D$. This immediately implies that L is big. Moreover, a simple integration shows that $L^k \cdot V > 0$ for any k -dimensional subvariety which is not entirely contained in D . By Theorem 3.11 we conclude that $\mathbf{B}_+(L)$ is contained in D . It then follows that L is ample modulo D .

Fact 5.6. *Let L be semi-ample and strictly nef modulo D . Then L is ample modulo D .*

Let us construct examples of line bundles which are ample modulo a divisor. A wealth of examples can be constructed using the following.

Fact 5.7. *Let X be a smooth projective variety and let D be a simple normal crossing divisor such that $K_X + \alpha D$ is ample for some $\alpha \in [0, 1)$. Then $K_X + D$ is ample modulo D .*

We now want to give examples of pairs where $K_X + D$ is ample modulo D but such that $K_X + \alpha D$ is never ample for any $\alpha \in \mathbb{R}$.

Example 5.8. *Let X be a smooth projective surface and let D be a semi-stable curve. Let assume (X, D) to be D -minimal, of log-general type, without interior (-2) -curves. Moreover, assume the existence of at least one (-2) -curve in D , denoted by E , such that $E \cdot (D - E) = 2$. Note that $K_X + D$ is semi-ample since the abundance conjecture holds in dimension two. Moreover, $K_X + D$ is strictly nef modulo D but it is not ample since $(K_X + D) \cdot E = 0$. By Fact 5.6 we conclude that $K_X + D$ is ample modulo D . On the other hand $(K_X + \alpha D) \cdot E = 0$ constantly in α .*

For more details regarding Example 5.8 we refer to Section 3 in [DiC12b]. If the reader is interested in explicitly constructing a pair as in Example 5.8 we refer to Example 5.13 below.

Now, in all the above examples $r(X, D) < 1$. Let us construct a pair (X, D) with $K_X + D$ big, semi-ample and strictly nef modulo D such that $r(X, D) = 1$.

Example 5.9. *Let X a smooth projective surface and D a reduced divisor consisting of two components D_1 and D_2 meeting transversally at a point p . Assume $K_X + D$ to be ample. Let $\pi : Y \rightarrow X$ be the blow up at $p \in D$. Let D' be the union of the strict transforms of D_1 and D_2 in Y and the exceptional divisor E . Then $K_Y + D' = \pi^*(K_X + D)$. We conclude that $K_Y + D'$ is big, nef and strictly nef modulo D' . Then $K_Y + D'$ is ample modulo D' . On the other hand*

$$(K_Y + \alpha D') \cdot E = (\alpha - 1) < 0$$

which therefore implies $r(X, D) = 1$.

Unfortunately, as shown in Example 5.9, being nef and ample modulo a divisor is not an open condition even for divisors of the form $K_X + D$. Nevertheless, we have the following.

Theorem 5.10. *Let X be a smooth projective variety and let D be an effective reduced divisor such that $K_X + D$ is big, nef and strictly nef modulo D . Assume that $r(X, D) < 1$. Then $K_X + \alpha D$ is big, nef and ample modulo D for any $\alpha \in (\frac{n+1}{n+2}, 1)$.*

Proof. Since we are assuming $r(X, D) < 1$, by Corollary 4.13 we know that

$$\tau(X, D) \leq r(X, D) \leq \frac{n+1}{n+2}.$$

It then remains to check the ampleness modulo D . As in the proof of Proposition 4.12, observe that given an irreducible curve C such that $(K_X + D) \cdot C > 0$ then $(K_X + \alpha D) \cdot C > 0$ for any $\alpha \in (\frac{n+1}{n+2}, 1]$. By Theorem 3.7, for any $\alpha \in (\frac{n+1}{n+2}, 1)$ we have that $K_X + \alpha D$ is semi-ample. By Fact 5.6 the proof is then complete. \square

Note that the strategy behind the proof of Theorem 5.10 can be used to prove the following.

Corollary 5.11. *Let (X, D) be such that $K_X + D$ is big, nef and strictly nef modulo D . Assume there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. Then, for any $\alpha \in (\frac{n+1}{n+2}, 1)$, the \mathbb{R} -cohomology class of $K_X + \alpha D$ can be represented by a smooth closed $(1, 1)$ -form which is everywhere positive semi-definite and strictly positive outside D .*

Proof. By Proposition 4.12 we have that $r(X, D) < 1$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. \square

We decided to explicitly state Corollary 5.11 because it has nice applications in the theory of negatively curved Kähler-Einstein metrics on quasi-projective varieties. These applications rely on some recent advancements in the theory of degenerate complex Monge-Ampère equations with singular right hand side [DP06]. Thus, let (X, D) be as in Corollary 5.11. For any $\alpha \in (\frac{n+1}{n+2}, 1)$ denote by γ_α the smooth semi-positive form representing the cohomology class of $K_X + \alpha D$. Given a smooth volume form Ω on X and Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}_X(D_i)$, consider the family of degenerate complex Monge-Ampère equations

$$(6) \quad (\gamma_\alpha + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^\varphi \frac{\Omega}{\prod_i \|\sigma_i\|^{2\alpha}}$$

where by D_i and σ_i we respectively denote the irreducible components of D and the associated defining sections. By Theorem 6.1. in [DP06], for a fixed α this equation admits a unique solution $\varphi \in L^\infty(X)$ which is smooth on $X \setminus D$. Moreover, by appropriately choosing Ω and the $\|\cdot\|_i$ in 6 we can arrange γ_φ^α to be Einstein with negative scalar curvature on $X \setminus D$. To sum up, given a pair (X, D) such that $K_X + D$ is big, strictly nef modulo D and with $r(X, D) < 1$ then the Kähler-Einstein theory on $X \setminus D$ is rather well-understood. Of course, the asymptotic behavior of γ_φ^α along D as well as the limiting behavior of this family as $\alpha \rightarrow 1$ remain quite mysterious.

Remark 5.12. *The 1-parameter family of negatively curved Kähler-Einstein metrics γ_φ^α cannot represent ample cohomology classes unless there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$. Recall that in this case*

the form γ_α can be chosen to be a Kähler metric. Compare Proposition 2.10 and Theorem 4.15. In this case the asymptotic behavior of γ_φ^α along D is much better understood. In fact, thanks to the recent works [JMR11] and [MR12], we know that for any $\alpha \in (\frac{n+1}{n+2}, 1)$ the Kähler-Einstein current γ_φ^α has edge singularities of cone angle $2\pi(1-\alpha)$ along D .

Let us conclude this section by giving an example of a pair (X, D) which admits both complete and incomplete negatively curved Kähler-Einstein metrics on $X \setminus D$, but such that $K_X + \alpha D$ is never ample for any $\alpha \in \mathbb{R}$.

Remark 5.13. Let $\mathcal{H} \times \mathcal{H}$ be the product of two copies of the upper half plane of \mathbb{C} . Let Γ be a discrete torsion free non-uniform subgroup of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Denote by X^* the Baily-Borel compactification [AMRT10] of $(\mathcal{H} \times \mathcal{H})/\Gamma$. Let X be the minimal resolution of X^* . By appropriately choosing Γ , see [Hir71] for beautiful explicit constructions, the resolution of the cusp points in X^* is given by a divisor D whose connected components are cycles of smooth rational curves with negative self-intersection. More precisely, for any of the irreducible components we have $D_i^2 \leq -2$. Furthermore, we have at least one (-2) -curve in D while there are no (-2) -curves in $X \setminus D$ since the curvature of $\mathcal{H} \times \mathcal{H}$ is nonpositive. Now, let D_i be a (-2) -curve in D . We then have that $(K_X + \alpha D) \cdot D_i = 0$ constantly in α . On the other hand $X \setminus D$ admits a complete nonpositively curved Kähler-Einstein metric coming from $\mathcal{H} \times \mathcal{H}$. Moreover, the pair (X, D) satisfies the hypothesis of Corollary 5.11, see Example 5.8. Thus, for $\alpha \in (\frac{3}{4}, 1)$, $X \setminus D$ also admits a 1-parameter family of incomplete negatively curved Kähler-Einstein metrics γ_φ^α . Of course, this family has not edge singularities along D in the sense of Definition 5.

6. POSITIVITY FOR FANO AND LOG-FANO VARIETIES

In this section we study the existence of uniform bound for the nef threshold when X is a Fano variety or a log-Fano variety, i.e. $-K_X$ or $-(K_X + D)$ is ample.

Question 6.1. Let X be a smooth variety and let D an effective divisor with $-(K_X + D)$ ample. Is there a fixed $\alpha_n < 1$, which depends only on n , such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_n, 1]$?

It is quite tempting to do the same thing as we did in the previous section. For example if we apply Lemma 4.3 we get that $-(K_X + \frac{n+1}{n}D)$ is nef. Since we want to deal with $\alpha < 1$ we do not get any useful information. Probably the best reason that explains why we cannot do the same thing is that the above question has negative solution.

Example 6.2. We follow the notation in [Har77]. Let $C = \mathbb{P}^1$ and $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{O}_C(-e)$ where e is a positive integer. We define $X_e := \mathbb{P}(\mathcal{E})$, it is a rational ruled surface. By Corollary V.2.11 in [Har77] we know that $K_{X_e} \equiv -2C_0 - (e+2)f$ where C_0 is the zero section and f is a general fibre. Let $D := C_0$. Then $-K_{X_e} - D = C_0 + (e+2)f$ is an ample divisor by Corollary V.2.18 in [Har77]. Then by the same corollary $-K_{X_e} - \alpha D = (2-\alpha)C_0 + (e+2)f$ is ample if and only if $\alpha > \frac{e-2}{e}$. Then if e goes to infinity α must be arbitrarily close to one. Note that $(-(K_{X_e} + D))^2 = e+4$.

We would like to understand how α depends on the pair (X, D) . We will prove that the situation in Example 6.2 is the most general and that α depends only on the dimension of n and the top self-intersection of $-(K_X + D)$. We need two preliminary results. The first one is an effective version of Kollár-Matsusaka theorem due to Demailly and Siu. We follow the presentation in [Laz04b].

Theorem 6.3 (Kollár-Matsusaka, Demailly, Siu). *Let X be a smooth projective variety. Let L be an ample divisor and B be a nef divisor. Then for any $m \geq M$ the divisor $mL - B$ is very ample, where*

$$M = (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1} \cdot (B + H))^{(3^{n-1}+1)/2} (L^{n-1} \cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)-1/4}},$$

and $H = (n^3 - n^2 - n - 1)(K_X + (n + 2)L)$.

The second result is a theorem of M^cKernan and it is an important step in the study of boundness of singular Fano varieties, see [M^cK02] and Corollary 1.8 in [HMX12].

Theorem 6.4 (M^cKernan). *Fix r and n integers. Then there is a real number M such that if we have a pair (X, D) where*

- (1) X has dimension n ,
- (2) $-(K_X + D)$ is big and nef,
- (3) (X, D) is klt and
- (4) $r(K_X + D)$ is Cartier

then $-(K_X + D))^n \leq M$.

We can now prove Theorem 1.3 as stated in the Introduction 1.

Proof of Theorem 1.3. Suppose that there exists M_0 as in the statement. If we prove that there exists a positive integer M such that $-K_X + M(-K_X - D)$ is ample for any $(X, D) \in \mathfrak{B}$ then we have done, because it implies that we can take $\alpha_0 = \frac{M}{M+1}$. In order to prove the existence of such an integer we want to apply effective Kollár-Matsusaka. Write $L := -(K_X + D)$ which is ample by hypothesis. Then if we define $B = K_X + (n + 1)L$ by Lemma 4.3, we know that B is nef. For the sake of clarity we do not keep track of all the constants which appears in Theorem 6.3. In order to get an integer M which depends only on L^n we need to bound all the intersection numbers appearing in the formula with L^n . We can write $B = nL - D$ and $H = c(n)((n + 1)L - D)$ where $c(n) = n^3 - n^2 - n - 1$. Since L is ample and D is effective we get the following inequalities

$$L^{n-1} \cdot B < nL^n \quad \text{and} \quad L^{n-1} \cdot H < c(n)(n + 1)L^n.$$

Plugging them in the formula of Theorem 6.3 we get that there are positive constants $a(n)$ and $b(n)$ depending only on n such that for any $m \geq a(n)(L^n)^{b(n)}$ the divisor $mL - B$ is ample. This is exactly what we wanted at the beginning of the proof.

On the other hand suppose that we can find α_0 as in (1). Choose a rational number $\alpha \in (\alpha_0, 1)$ and let r be its denominator. For any pair $(X, D) \in \mathfrak{B}$ we have

that $(X, \alpha D)$ is a klt pair and $r(K_X + \alpha D)$ is Cartier. Then Theorem 6.4 applies and we get that there exists M_0 such that $-(K_X + \alpha D)^n \leq M_0$ for any $(X, D) \in \mathfrak{B}$. Since $-K_X - D + (1 - \alpha)D = -K_X - \alpha D$ we get that also $-(K_X + D)^n$ is uniformly bounded. \square

The following is a corollary of the first part of the proof of the previous theorem.

Corollary 6.5. *Fix a positive integer n . Let X be a smooth projective variety of dimension n and let D be an effective divisor such that $-(K_X + D)$ is ample. Then there are positive integers $a(n)$ and $b(n)$, depending only on n , such that*

$$-\left(K_X + \frac{\alpha}{\alpha + 1}D\right)$$

is ample for any $\alpha \geq a(n)((-K_X - D)^n)^{b(n)}$. Furthermore $a(n)$ and $b(n)$ are explicitly computable functions of n .

In order to state our next corollary we need a preliminary definition.

Definition 6.6. *Let X be an irreducible projective variety and L be a Cartier divisor. Then the volume of L is defined as follow*

$$\text{vol}(L) := \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mL))}{m^n/n!}.$$

By definition $\text{vol}(L) > 0$ if and only if L is big. If L is big and nef then $\text{vol}(L) = L^n$ but in general it is very hard to compute the volume of a divisor.

Corollary 6.7. *Let \mathfrak{B} be a set of pairs (X, D) such that X is a smooth projective variety of dimension n and D is a reduced effective divisor with simple normal crossing support such that $-(K_X + D)$ is ample. Suppose there exists a positive integer M such that $\text{vol}(-K_X) \leq M$ for any $X \in \mathfrak{B}$. Then there exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_0, 1]$ and any $(X, D) \in \mathfrak{B}$.*

Proof. It simply follows from Theorem 1.3 and the fact that $-(K_X + D)^n \leq \text{vol}(-K_X)$. \square

The above corollary tells us that we can choose α_0 to be independent of D .

Let us compute the volume of the anti-canonical divisors in Example 6.2. We will use the following theorem of Zariski. See [Laz04a] for details.

Theorem 6.8 (Zariski decomposition). *Let X be a smooth projective surface and let L be a pseudo-effective divisor. Then L can be written uniquely as a sum $L = P + N$ of \mathbb{Q} -divisors with the following properties:*

- (1) P is a nef divisor;
- (2) $N = \sum a_i E_i$ is effective, and if $N \neq 0$ then the intersection matrix $(E_i \cdot E_j)_{i,j}$ is negative definite.
- (3) $P \cdot E_i = 0$ for any component E_i of N .

Furthermore for every $m \geq 1$

$$H^0(X, \mathcal{O}_X(mL - \lceil mN \rceil)) = H^0(X, \mathcal{O}_X(mL)).$$

P and N are called respectively the positive and negative parts of L . Note that the last statement of Theorem 6.8 tells us that $\text{vol}(L) = P^2$.

Example 6.9. Let $X := X_e$ and D be as in the previous example. We can assume $e \geq 3$ otherwise $-K_X$ is big and nef. Since $-K_X$ is big we can consider the Zariski decomposition $-K_X = P + N$. In particular $\text{vol}(-K_X) = P^2$. It is easy to see that

$$-K_X = \left(1 + \frac{2}{e}\right) (C_0 + ef) + \left(1 - \frac{2}{e}\right) C_0$$

is the Zariski decomposition of $-K_X$. Then

$$\text{vol}(-K_X) = P^2 = \left(1 + \frac{2}{e}\right)^2 (C_0 + ef)^2 = (e + 4) + \frac{4}{e}.$$

We now deal with the case $-K_X$ ample. The main question is the following.

Question 6.10. Let X be a smooth variety with $-K_X$ ample and let D be an effective divisor. Is there a uniform α_n , which depends only on n , such that $-(K_X + \alpha D)$ is ample for any $\alpha \in [0, \alpha_n)$?

It is easy to see that in this case the answer is no even if we fix X . Recall that in the previous case if we fix X we can find a uniform bound that works for any divisor on X because it is enough to bound $\text{vol}(-K_X)$.

Example 6.11. Let $X = \mathbb{P}^n$ and $D = \mathcal{O}_X(d)$ for some $d > 0$. Then $-(K_X + \alpha D)$ is ample if and only if $\alpha < (n + 1)/d$. Then if d is arbitrarily large we need to take α arbitrarily small.

In order to settle Question 6.10 we need a well known result of Kollár, Miyaoka and Mori. See Theorem 4.4 in [KMM92].

Theorem 6.12 (Kollár-Miyaoka-Mori). Fix an integer n . Let X be a smooth Fano variety of dimension n . Then

$$(-K_X)^n \leq (n + 1)^n n^{(2^n - 1)n(n + 1)} \left(1 + \frac{1}{n} + \frac{n + 1}{n(n - 1)}\right)^n.$$

In particular the set of smooth Fano varieties of dimension n forms a bounded family.

Finally, we give the proof of the last theorem stated in the Introduction 1.

Proof of Theorem 1.4. Suppose that there exists M_0 as in the statement. We want to apply Kollár-Matsusaka's theorem to find an integer M such that $-K_X - D + M(-K_X)$ is ample. Let $L := -K_X$ and let $B := K_X + D + 2nL$. We know by the cone theorem that B is nef. We need to control the following intersection numbers $B \cdot L^{n-1}$ and $H \cdot L^{n-1}$ where H is as in the statement of Theorem 6.3. We can write $B = (2n - 1)L + D$ and $H = c(n)(n + 1)L$ where $c(n) = n^3 - n^2 - n - 1$. Since L is ample we have that $L^n \geq 1$ and Theorem 6.12 tells us that there exists a function $f(n)$ of n such that $L^n \leq f(n)$. Then we can find the desired integer M applying Theorem 6.3.

Now assume that there exists α_0 as in (1). By assumption we know that $H := -(K_X + \alpha_0 D)$ is a nef \mathbb{R} -divisor. Then

$$\alpha_0 D \cdot (-K_X)^{n-1} = (-K_X - H) \cdot (-K_X)^{n-1} \leq (-K_X)^n,$$

where the last inequality follows from the fact that H is nef and $-K_X$ is ample. Then it follows from Theorem 6.12 that we can uniformly bound $D \cdot (-K_X)^{n-1}$. \square

The first part of the proof gives the following.

Corollary 6.13. *Let (X, D) be a pair such that X is a smooth Fano variety of dimension n and D is a reduced effective divisor with simple normal crossing support. Then there are two positive integers $a(n)$ and $b(n)$ depending only on n such that $-(K_X + \frac{1}{\alpha+1}D)$ is ample for any $\alpha \leq a(n) (-D \cdot K_X^{n-1})^{b(n)}$.*

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